Journal of Chemical Technology and Metallurgy, 59, 2, 2024, 457-464

COVERING A MAXIMUM NUMBER OF POINTS BY A FIXED NUMBER OF EQUAL DISKS VIA SIMULATED ANNEALING

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Received 09 July 2023
Accepted 04 January 2024
DOI: 10.59957/jctm.v59.i2.2024.26

ABSTRACT

The presented paper considers the problem of covering a maximum number of n given points in the plane by m equal disks of radius r. A point is covered if it is inside one or more than one disk. The disks need to be placed in the plane in such a way that a maximum number of points are covered. To solve the problem, an objective function, called energy, is introduced in such a way that the greater the covering is, the lower the energy is. Thus, a configuration of disks with minimum energy is a configuration with maximum covering. To find a configuration of disks that minimizes the energy, a stochastic algorithm based on the Monte Carlo simulated annealing technique is proposed. The algorithm overcomes potential local minima, which, as shown in the paper, are quite likely to occur. The computational complexity of the algorithm is O(mn). The algorithm is tested on several cases demonstrating its efficiency in finding global minima of the energy, i.e. configurations with maximum covering.

Keywords: *disk cover problem, continuous optimization, stochastic algorithm, Monte Carlo simulation.*

INTRODUCTION

This work considers the following problem. Let n points be given in the plane. One needs to place m disks of radius r in such a way that a maximum number of the given points are covered. A point is covered if it lies inside one or more than one disk. The considered problem can be used as a basis to model many real life problems. For example, how to place m radio transmitters, each capable of covering a circular area of radius r, so that a maximum number of locations, e.g. towns, villages, etc., get covered. Another example could be the positioning of an m anti-missile defense systems of range r so that a maximum number of places are protected.

The considered problem is a variant of the unit disc cover problem (UDC) [1 - 3] and the minimum geometric disk cover problem (MGDC) [4, 5]. Such problems are known to be NP-complete [6]. For this reason, in the literature, mainly algorithms that find only approximate solutions are available [3, 5]. This work proposes a Monte Carlo algorithm that is capable of finding, with high degree of certainty, exact solutions of the problem. The computational complexity of the proposed algorithm is $O(mn)$.

In Fig. 1 you can see ten points and two disks of radius r. The disks are placed in such a way that seven points are covered. This is the maximum number of points that can be covered by two disks of radius r. However, this is not the only possible way to cover seven points by two disks of radius r. Since the points that are covered by a particular disk lie strictly inside this disk, the disk can be displaced by an arbitrary but small enough displacement getting another way of covering seven points, i.e. another solution. Hence, there are infinitely many solutions. Note that the answer to the question "what is the maximum number of points that can be covered" is unique, and in this case is "seven". However, the number of solutions, i.e. the different ways to cover a maximum number of points, is infinite.

The goal of this work is to develop an algorithm that places/arranges the disks in such a way that a maximum number of points are covered. Note that solving this problem automatically answers the question of what is the maximum number of points that can be covered.

Our approach to solving the problem is to introduce an objective function, called energy of the system, in the following way. The energy of a particular state of the system, i.e. a particular configuration of disks, is the sum of all uncovered point. That is, if there is one uncovered point, the energy is one, if there are two uncovered points, the energy is two, etc. Hence, a state (configuration) with a minimum energy is a state with a maximum number of covered points. Thus, the original problem reduces to finding a state with minimum energy, which is on optimization problem [7 - 10]. Since the optimization variables, namely the centers of the disks, are continuous variables, the problem is a continuous optimization problem. There are many powerful methods for continuous optimization but not all of them are appropriate for the solution of our problem. Since, as shown, the system may have many local minima, methods such as the method of steepest descent [11, 12] and other methods that make steps only in the direction of decreasing the energy, are not appropriate. Such methods, instead of getting the global minimum, may well get stuck in any of the local minima.

To solve the optimization problem, a stochastic algorithm [13] based on the simulated annealing technique is proposed [14 - 20]. The simulated annealing is an importance sampling Monte Carlo method [21 - 23]. The proposed algorithm overcomes local minima and, as demonstrated, successfully finds global minima of the system.

EXPERIMENTAL

Mathematical formulation of the problem

Let n points P_i, $i = 1,2,\ldots,n$ with Cartesian coordinates (x_i, y_i) be given in the plane. In addition, a radius r is given and a positive integer number m (number of disks). Consider a disk C_j of radius r and center O_j. The Cartesian coordinates of O_j are $(\tilde{x}_j, \tilde{y}_j)$. The distance between the point P_i and the center O_j is

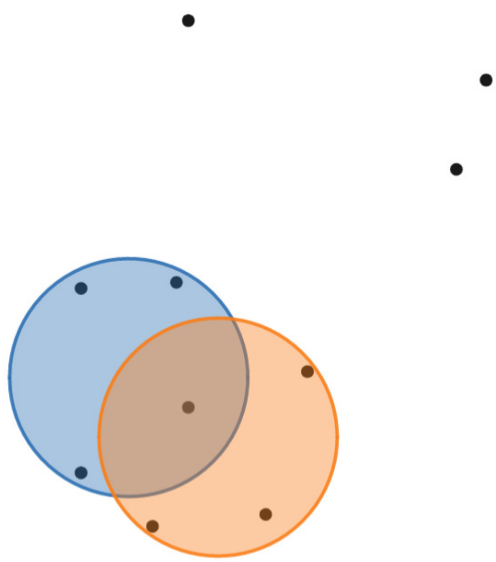

Fig. 1. Ten points and two disks of radius r. The disks cover a maximum number of points.

$$r_{ij} = \sqrt{\left(x_i - \tilde{x}_j\right)^2 + \left(y_i - \tilde{y}_j\right)^2} \quad (1)$$

The point P_i is inside the disk C_j if $r_{ij} < r$ (2)

If the point P_j is inside the disk C_j, then the disk covers the point. If the point is covered by one or more than one disk, then the point is covered. Otherwise the point is not covered. Total of m disks of radius r need to be placed in such a way that the number of points that are covered is maximum. Note that any arrangement/configuration of disks is uniquely determined by the centers O_j, $j = 1,2,\ldots,m$ of the disks. Since all disks have the same radius, permuting the disks does not change the configuration.

Let P be a point in the plane and ω be a configuration of disks. The energy of the point P is defined as

$$e(P,\omega) = \begin{cases} 0, P \in \omega \\ 1, P \notin \omega \end{cases} \quad (3)$$

where $P \in \omega$ means that P is covered by ω, while $P \notin \omega$ means that P is not covered by ω. Thus, a point has energy zero if it is covered by the configuration of disks ω and one otherwise. The energy of the whole system is defined as the sum of the energy of all points:

$$E(\omega) = \sum_{i=1}^{n} e(P_i, \omega). \quad (4)$$

Let Ω be the configuration space, i.e. the set of all possible configurations of disks. Let E_{min} be the global minimum of the energy, i.e.

$$E_{min} = \min_{\omega \in \Omega} E(\omega). \quad (5)$$

Now the problem under consideration can be formulated in the following way. Find a configuration of disks α such that

$$E(\alpha)=E_{min}, \quad (6)$$

The configuration α minimizes the energy and therefore it maximizes the number of covered points. It is a solution to our problem, but it is not the only solution. However, the number of points covered by α is the maximum number of points that can be covered by m disks of radius r.

Formulated in this way the problem is an unconstrained continuous optimization problem for finding the global minimum of the objective function E called energy of the system. As discussed, some of the existing optimization methods are not appropriate for the solution of this problem because the objective function E can have local minima.

Existence of local minima

This section provides a constructive proof for the existence of local minima of the energy E. Consider the configuration shown in Fig. 2. The system has eleven points and two disks of radius r. Since three points are not covered, the energy of the system is three. Note that moving the right disk to any position will either increase the energy or not change it. The same applies to the left disk. Therefore, the state (configuration of disks) shown in Fig. 2 is a local minimum on the simulation graph. In other words, all neighboring states, that is states that can be reached in one step of the algorithm, have energy greater than or equal to the energy of this state. This holds for all possible algorithms for which only one disk is allowed to be moved at a time. The state in Fig. 3, however, covers nine points with two disks of radius r. This state is a global minimum and its energy is two. It can be reached from the local minimum in two steps. One way to reach it is first to move either of the disks in Fig. 2 to cover the three uncovered points and then to move the other disk to cover the six middle points. The first move results in an increase of energy.

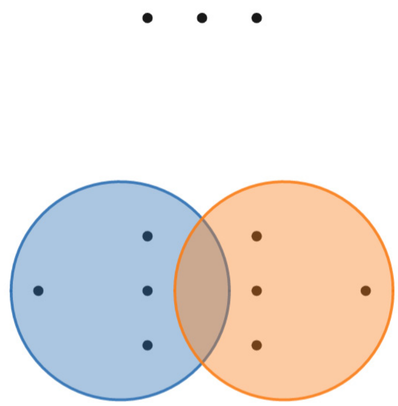

Fig. 2. Local minimum, $E = 3$.

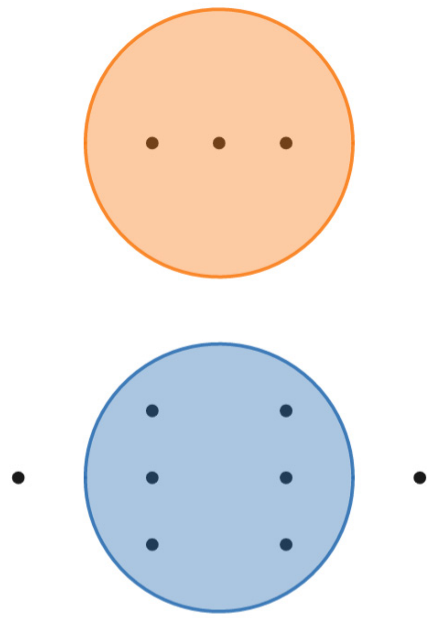

Fig. 3. Global minimum, $E = 2$.

The other way to reach the global minimum is first to move either of the disks in Fig. 2 to cover the six middle points and then cover the three top points by the other disk. In this case, too, the first move increases the energy of the system. Thus, if a naïve optimization simulation algorithm that always makes steps in the direction of decreasing the energy is used, then reaching the state in Fig. 2 will result in getting stuck in this state. Therefore, an optimization method that is capable of finding global minima in the possible presence of local minima is needed. One such method is the simulated annealing technique.

Simulated annealing algorithm

To solve the energy minimization problem, the following simulated annealing algorithm is proposed. First, the lowest and highest x and y coordinates of the given points are determined and a rectangular domain D in the xOy plane where the disks could be placed is defined. A disk outside the domain D cannot cover any of the given points because it is too far. Hence, the simulation is restricted to placing disks inside the domain D only. Next, an initial state, i.e. a configuration of disks lying in D, is chosen at random and its energy is calculated. Then, the following Monte Carlo simulation is performed. A disk is chosen at random. Then, a new positon is chosen at random. The change of energy ΔE resulting from moving the disk from its old position to its new position is calculated. If $\Delta E < 0$, the move is accepted. If $\Delta E \geq 0$, the move is accepted with probability

$$Prob = \exp(-\Delta E/T), \quad (7)$$

where T is a parameter called temperature. The probability of rejecting the move is $1 - Prob$. The higher the increase ΔE of the energy, the greater the probability of rejecting the move. Note that decreasing the temperature T (called cooling) increases the probability of rejecting the move. If the move is accepted, the disk is moved to its new position and the energy of the new state is computed by adding ΔE to E. Then, again, a disk is and new position are chosen at random and an attempt is made to move the disk. Thus, a certain number of attempts are made, e.g. 1000 or more, and anytime the move is accepted the chosen disk is moved to its new position and the energy E is changed accordingly. Then, the temperature is lowered according to some cooling schedule, e.g. T becomes 2/3 of its current value, and the whole process is repeated. When the system has been cooled down enough, then, supposedly, the global minimum has been reached. The proposed algorithm is presented schematically in Fig. 4.

One of the main features of the proposed simulated annealing algorithm is that it allows moves that increase the energy. Thus, the system cannot get stuck in a local minimum, or is highly unlikely to do so. Of course, the successful reaching of the global minimum depends on the proper choice of the simulation parameters and the cooling schedule. The greater the number of attempts and the slower the process of cooling, the greater the

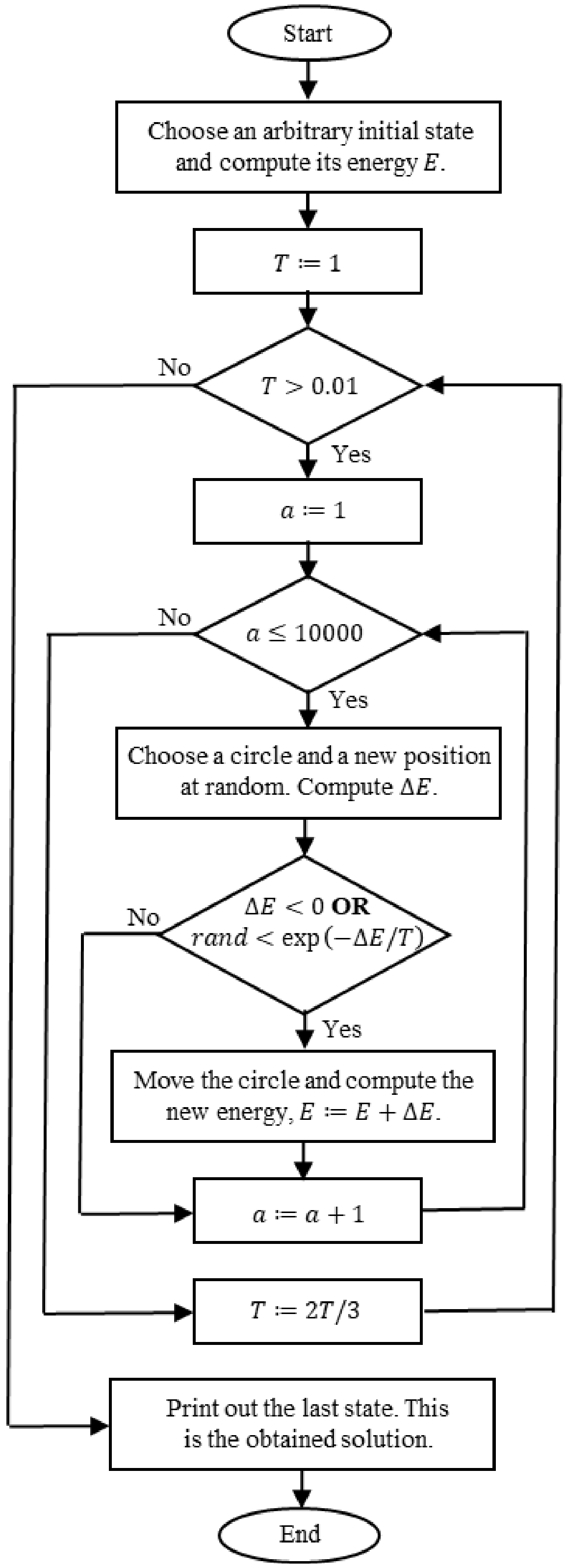

Fig. 4. The proposed simulated annealing algorithm. In the algorithm *rand* is a random number between 0 and 1.

probability that the final state reached in the simulation is the global minimum. One should also provide that the initial temperature is high enough and the final temperature is low enough.

Provided that the area of the allowed domain *D* increases linearly with the number of points *n*, the number of moves that need to be made in order to sample efficiently the domain *D* with each of the *m* disks must be proportional to *m*. Hence, the computational complexity of the proposed algorithm is *O(mn)*.

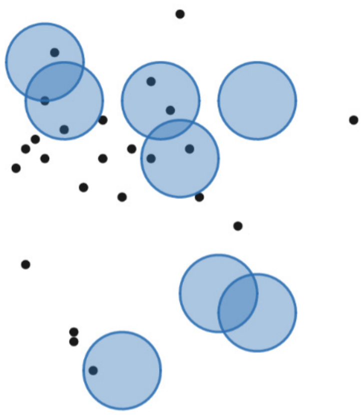

Fig. 5. Initial state A, *E = 16*.

RESULTS AND DISCUSSION

This section presents results obtained by the proposed stimulated annealing algorithm. The set of 24 points shown in Fig. 5 is considered. First, in order to cover the point, 8 disks of radius 2 are used. One of the randomly generated initial states that is used is shown in Fig. 5. It is called initial state A and its energy is 16, that is, the initial number of uncovered points is 16. A number of simulations are performed using state A as initial state and each time a final state with energy 2, i.e. 2 uncovered points, is obtained. Two of the final states that were obtained are shown in Fig. 6 and Fig. 7. Likewise, for initial state B shown in Fig. 8, a number of simulations were performed and again final states with energy 2 were obtained. Two of these states are shown in Fig. 9 and Fig. 10. For any other arbitrarily generated initial state that was tried, finial states with energy 2 were obtained. Hence, *E = 2* is the minimum energy that a configuration of 8 disks of radius 2 can have. Therefore, the maximum number of points that can be covered is 22. The configurations of disks shown in Fig. 6, Fig. 7, Fig. 9, and Fig. 10 all cover 22 points, hence, all of them are solutions to the problem of maximal covering of the given 24 points by 8 disks of radius 2.

Results for 5 disks of radius 2 are shown in Fig. 11 - Fig. 13. Obviously, the minimum of the energy is *E=5* and the maximum number of points that can be covered is 19. How the energy changes with temperature for the considered cases is shown in Fig. 14 - Fig. 16. Note that the temperature decreases as $T = (2/3)^k$; $k = 0, 1, ..., 12$. In the figures, *E* is shown as a function of *k*.

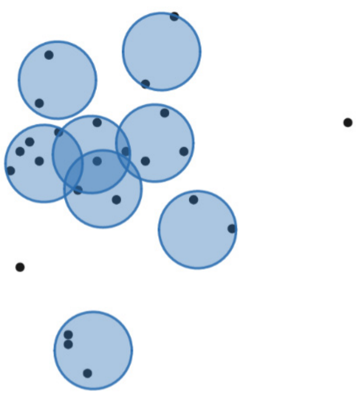

Fig. 6. Solution A-1, *E* = 2.

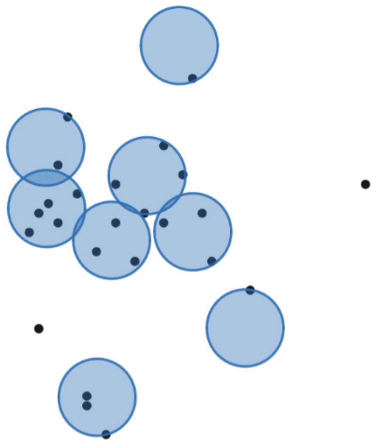

Fig. 7. Solution A-2, *E = 2*.

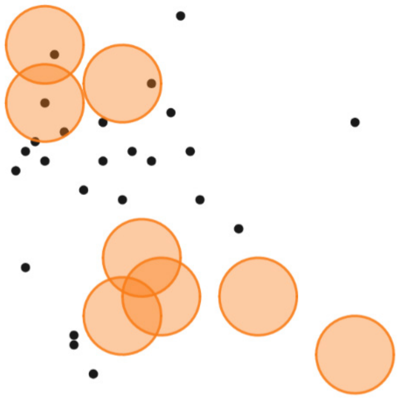

Fig. 8. Initial state B, *E = 20*.

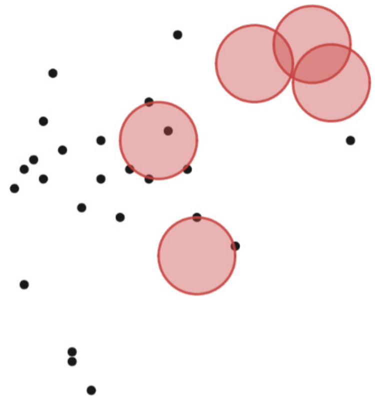

Fig. 11. Initial state C, *E = 20*.

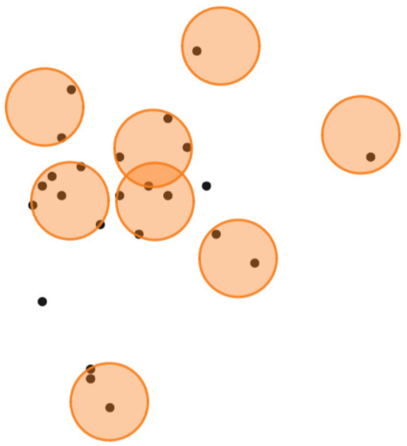

Fig. 9. Solution B-1, *E = 2*.

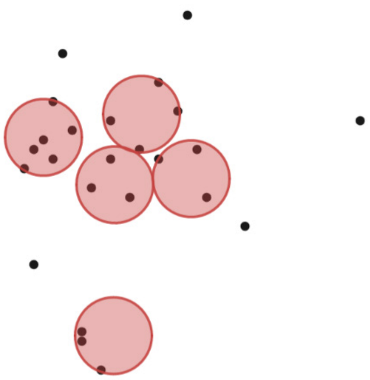

Fig. 12. Solution C-1, *E = 5*

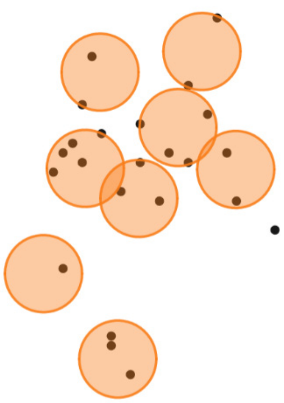

Fig. 10. Solution B-2, *E = 2*

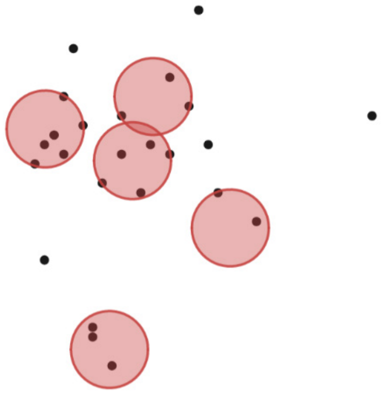

Fig. 13. Solution C-2, *E = 5*.

CONCLUSIONS

This paper considered the problem of covering a maximum number of given points by a fixed number of equal disks. An objective function called energy was introduced and the problem was converted to a continuous optimization problem for minimizing the energy. For the solution of the optimization problem a simulated annealing algorithm was developed. It was demonstrated that the algorithm successfully finds global minima of the energy overcoming potential local minima. The global minima correspond to configurations of disks that cover a maximum number of points, hence they are the sought solutions of the original problem.

Acknowledgements

The research is funded by the Sofia University Research Fund through Contract No. 80-10-57/25.04.2023.

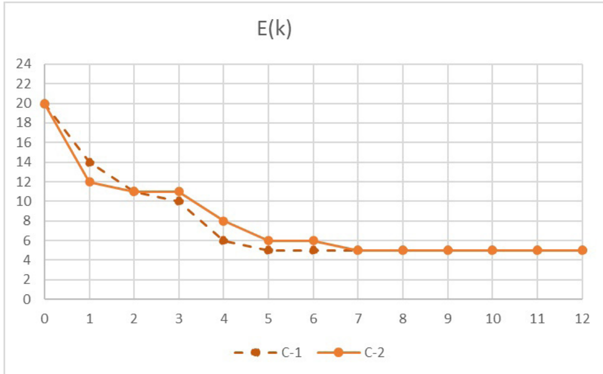

Fig. 14. E vs k for initial state A and solutions A-1 and A-2.

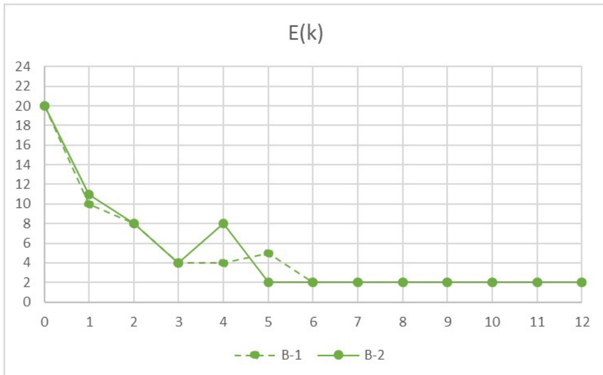

Fig. 15. E vs k for initial state B and solutions B-1 and B-2.

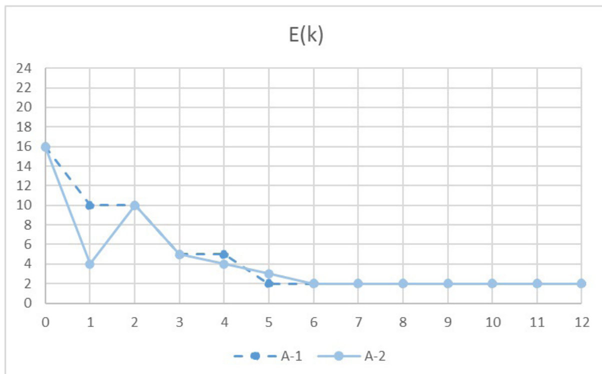

Fig. 16. E vs k for initial state C and solutions C-1 and C-2.

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